

Partial Information Differential Games for Mean-Field SDEs

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Abstract: This paper is concerned with non-zero sum differential games of mean-field stochastic differential equations with partial information and convex control domain. First, applying the classical convex variations, we obtain stochastic maximum principle for Nash equilibrium points. Subsequently, under additional assumptions, verification theorem for Nash equilibrium points is also derived. Finally, as an application, a linear quadratic example is discussed. The unique Nash equilibrium point is represented in a feedback form of not only the optimal filtering but also expected value of the system state, throughout the solutions of the Riccati equations.

Key Words: Partial information, Mean-field games, Backward stochastic differential equations, Maximum principle, Verification theorem

1 Introduction

In this paper, we study partial information stochastic differential game problems in which system states are governed by stochastic differential equations (SDEs) of mean-field type, in the sense that the coefficients of the SDEs depend not only on the system states, but also on their expected values. Also, the SDEs of mean-field type are often used to describe the aggregate behavior of lots of mutually interacting particles at mesoscopic level and play an important role in physics, finance, economics, etc. For more information, we refer the reader, for instance, to [6, 12] as well as the references therein. Recently, a new kind of backward SDEs (BSDEs) of mean-field type has been studied by Buckdahn et al. [3, 4] which is called mean-field BSDEs. For classical control problems of SDEs without mean field, we refer the readers to [13, 16], etc.

Mean-field games and mean-field control problems have received considerable attention in the probability and optimal control literature in recent years. Li [11] studied the stochastic maximum principle for mean-field SDEs with convex control domain and also got the verification theorem under additional conditions. Buckdahn, Djehiche and Li [2] used the classical spike perturbation and derived a Peng-type general stochastic maximum principle. Yong [15] investigated linear-quadratic (LQ) optimal control problems for mean-field SDEs and a feedback representation was obtained for the optimal control. Lasry and Lions [10] presented three examples of mean-field approach to modelling in Economics and Finance, derived nonlinear mean-field SDEs and established their links with various fields of Analysis. More recent developments and their applications of mean-fields games of SDEs can be found in Bensoussan, Sung and Yam [1], Carmona, Delarue and Lachapelle [5], Guant [8], etc., and the references therein. Different from the above works, we consider two players non-zero sum differential games of mean-field SDEs with partial

information and convex control domain. The distinguishing feature is the information available to the two players is the sub-filtration of full information. The problem we study may cover many control and game problems of mean-field SDEs with complete information as special cases. The present work will also enrich the relevant theory of stochastic filtering.

The rest of this paper is organized as follows. In Section 2, we specify the problem considered. Section 3 is devoted to deriving the stochastic maximum principle and verification theorem for Nash equilibrium points. Finally, in Section 4, we solve an LQ example to explain our application. By introducing the systems of some Riccati equations and forward-backward stochastic filtering equations of mean-field type, we give the feedback representation for the unique Nash equilibrium point.

2 Formulation of Problem

Let $|x|$ denote the Euclidian norm of $x \in \mathbb{R}^n$ and $\langle x, y \rangle$ be the inner product of $x, y \in \mathbb{R}^n$. The transpose and Euclidian norm of a matrix $M = (m^{ij})_{1 \leq i \leq n, 1 \leq j \leq d} = (m^1, \dots, m^d) \in \mathbb{R}^{n \times d}$ are expressed as M^* and $|M| = \sqrt{\text{trace}(MM^*)}$, respectively. Similarly, $\langle M_1, M_2 \rangle = \text{trace}(M_1 M_2^*)$ with $M_1, M_2 \in \mathbb{R}^{n \times d}$. Let $T > 0$ be a fixed constant and C be a positive constant which can be different from line to line. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P})$ be a complete filtered probability space on which \mathcal{F}_t denotes a natural filtration generated by a standard Brownian motion (w_1, w_2) with values in $\mathbb{R}^{d_1+d_2}$.

We only consider the case of two players and define the admissible control set \mathcal{U}_i for Player i ($i = 1, 2$) by

$$\mathcal{U}_i = \left\{ v_i(\cdot) : v_i(\cdot) : [0, T] \times \Omega \longrightarrow U_i, \text{ is a } \mathcal{G}_t^i\text{-adapted process satisfying } \mathbb{E} \int_0^T v_i(t)^2 dt < \infty \right\}, \quad (1)$$

where U_i is a nonempty convex subset of \mathbb{R}^{r_i} , and $\mathcal{G}_t^i \subseteq \mathcal{F}_t$ denotes the information available to Player i . Every element

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of \mathcal{U}_i is called an open-loop admissible control for Player i on $[0, T]$ ($i = 1, 2$). And $\mathcal{U}_1 \times \mathcal{U}_2$ is called the set of open-loop admissible controls for the players. Unless otherwise stated, Player 1 controls v_1 and Player 2 controls v_2 .

In the following, we consider the controlled state equation of mean-field type

$$\begin{cases} dx^{v_1, v_2}(t) = f(t, x^{v_1, v_2}(t), \mathbb{E}x^{v_1, v_2}(t), v_1(t), v_2(t)) dt \\ \quad + \sigma_1(t, x^{v_1, v_2}(t), \mathbb{E}x^{v_1, v_2}(t), v_1(t), v_2(t)) dw_1(t) \\ \quad + \sigma_2(t, x^{v_1, v_2}(t), \mathbb{E}x^{v_1, v_2}(t), v_1(t), v_2(t)) dw_2(t), \\ x^{v_1, v_2}(0) = x_0, t \geq 0, \end{cases} \quad (2)$$

and the cost functional

$$\begin{aligned} J_i(v_1(\cdot), v_2(\cdot)) \\ = \mathbb{E} \left[\int_0^T l_i(t, x^{v_1, v_2}(t), \mathbb{E}x^{v_1, v_2}(t), v_1(t), v_2(t)) dt \right. \\ \left. + \varphi_i(x^{v_1, v_2}(T), \mathbb{E}x^{v_1, v_2}(T)) \right], \quad (3) \end{aligned}$$

where the mappings

$$\begin{aligned} f(t, x, \tilde{x}, v_1, v_2) : \Omega \times [0, T] \times \mathbb{R}^{n+n} \times U_1 \times U_2 &\rightarrow \mathbb{R}^n, \\ \sigma_1(t, x, \tilde{x}, v_1, v_2) : \Omega \times [0, T] \times \mathbb{R}^{n+n} \times U_1 \times U_2 &\rightarrow \mathbb{R}^{n \times d_1}, \\ \sigma_2(t, x, \tilde{x}, v_1, v_2) : \Omega \times [0, T] \times \mathbb{R}^{n+n} \times U_1 \times U_2 &\rightarrow \mathbb{R}^{n \times d_2}, \\ l_i(t, x, \tilde{x}, v_1, v_2) : \Omega \times [0, T] \times \mathbb{R}^{n+n} \times U_1 \times U_2 &\rightarrow \mathbb{R}, \\ \varphi_i(x, \tilde{x}) : \Omega \times \mathbb{R}^{n+n} &\rightarrow \mathbb{R}, \end{aligned}$$

satisfy the following assumptions:

- (A1) the coefficients f, σ_1 and σ_2 are \mathcal{F}_t -adapted and bounded by $C(1 + |x| + |\tilde{x}| + |v_1| + |v_2|)$. They are also continuously differentiable with respect to (x, \tilde{x}, v_1, v_2) and their partial derivatives are Lipschitz continuous and uniformly bounded.
- (A2) l_1 and l_2 are \mathcal{F}_t -adapted and continuously differentiable with respect to (x, \tilde{x}, v_1, v_2) . φ_1 and φ_2 are \mathcal{F}_T -measurable and continuously differentiable with respect to (x, \tilde{x}) . Moreover, their partial derivatives are Lipschitz continuous and bounded by $C(1 + |x| + |\tilde{x}| + |v_1| + |v_2|)$.

Our aim is to find $(u_1, u_2) \in \mathcal{U}_1 \times \mathcal{U}_2$ such that

$$\begin{cases} J_1(u_1(\cdot), u_2(\cdot)) \leq J_1(v_1(\cdot), u_2(\cdot)), \\ J_2(u_1(\cdot), u_2(\cdot)) \leq J_2(u_1(\cdot), v_2(\cdot)), \end{cases} \quad (4)$$

for all $(v_1, v_2) \in \mathcal{U}_1 \times \mathcal{U}_2$. We call (u_1, u_2) an open-loop Nash equilibrium point of the game problem (if it exists).

Since \mathcal{G}_t^1 and \mathcal{G}_t^2 are the sub-information of \mathcal{F}_t , it implies this is the partial information game problem. On the contrary, when $\mathcal{G}_t^1 = \mathcal{G}_t^2 = \mathcal{F}_t, t \in [0, T]$, it reduces to be a complete information case. So the problem (1)-(4) denotes the partial information nonzero-sum differential game problem of the mean-field-type SDEs. For simplicity, we denoted it by Problem (MF).

3 Nash Equilibrium Point

3.1 Necessary Conditions

In this subsection, we establish a necessary conditions for Nash equilibrium points of Problem (MF). Let us suppose now that $(u_1(\cdot), u_2(\cdot))$ is an equilibrium point with the corresponding optimal state $x(\cdot)$. Then we define the perturbed control as follows:

$$u_{\varepsilon_i}(t) = u_i(t) + \varepsilon_i(v_i(t) - u_i(t)), \quad (5)$$

where $\varepsilon_i > 0$ is sufficiently small and $v_i(t)$ is an arbitrary admissible control of Player i ($i = 1, 2$). Notice that U_i is convex, then for $0 \leq \varepsilon_i \leq 1, 0 \leq t \leq T$, it yields $u_{\varepsilon_i}(t) \in \mathcal{U}_i$. We denote by $x_{\varepsilon_1}(\cdot)$ (resp. $x_{\varepsilon_2}(\cdot)$) the state $x^{u_{\varepsilon_1}, u_2}$ (resp. $x^{u_1, u_{\varepsilon_2}}$) associated with $(u_{\varepsilon_1}(\cdot), u_2(\cdot))$ (resp. $(u_1(\cdot), u_{\varepsilon_2}(\cdot))$). For simplicity, we set $g(t) = g(t, x(t), \mathbb{E}x(t), u_1(t), u_2(t))$, $g = f, \sigma_1, \sigma_2, l_1, l_2$.

We introduce the following variational equations:

$$\begin{cases} dx^i(t) = [f_x(t)x^i(t) + f_{\tilde{x}}(t)\mathbb{E}x^i(t) \\ \quad + f_{v_i}(t)(v_i(t) - u_i(t))] dt \\ \quad + [\sigma_{1x}(t)x^i(t) + \sigma_{1\tilde{x}}(t)\mathbb{E}x^i(t) \\ \quad + \sigma_{1v_i}(t)(v_i(t) - u_i(t))] dw_1(t) \\ \quad + [\sigma_{2x}(t)x^i(t) + \sigma_{2\tilde{x}}(t)\mathbb{E}x^i(t) \\ \quad + \sigma_{2v_i}(t)(v_i(t) - u_i(t))] dw_2(t), \\ x^i(0) = 0, \quad i = 1, 2. \end{cases} \quad (6)$$

For $i = 1, 2$, we set

$$\begin{aligned} \bar{x}_{\varepsilon_i}(t) &= \varepsilon_i^{-1}(x_{\varepsilon_i}(t) - x(t)) - x^i(t), \\ \psi_{\varepsilon_1}(t) &= (x_{\varepsilon_1}(t), \mathbb{E}x_{\varepsilon_1}(t), u_{\varepsilon_1}(t), u_2(t)), \\ \psi_{\varepsilon_1}^\lambda(t) &= (x(t) + \lambda\varepsilon_1(x^1(t) + \bar{x}_{\varepsilon_1}(t)), \mathbb{E}x(t) \\ &\quad + \lambda\varepsilon_1\mathbb{E}(x^1(t) + \bar{x}_{\varepsilon_1}(t))), \\ \phi_{\varepsilon_1}^\lambda(t) &= (x(t) + \lambda\varepsilon_1(x^1(t) + \bar{x}_{\varepsilon_1}(t)), \mathbb{E}x(t) \\ &\quad + \lambda\varepsilon_1\mathbb{E}(x^1(t) + \bar{x}_{\varepsilon_1}(t)), u_1(t) \\ &\quad + \lambda\varepsilon_1(v^1(t) - u_1(t)), u_2(t)). \end{aligned}$$

Then by a similar method as shown in Li [11] and Hui and Xiao [9] with a minor modification, we have the following convergence result.

Lemma 3.1. *Under Assumption (A1), we have*

$$\begin{aligned} \lim_{\varepsilon_1 \rightarrow 0} \mathbb{E} \sup_{0 \leq t \leq T} |x_{\varepsilon_1}(t) - x(t)|^2 &= 0, \\ \lim_{\varepsilon_2 \rightarrow 0} \mathbb{E} \sup_{0 \leq t \leq T} |x_{\varepsilon_2}(t) - x(t)|^2 &= 0. \end{aligned} \quad (7)$$

Proof. By Assumption (A1) and the Burkholder-Davis-

Gundy inequality, we derive

$$\begin{aligned}
& \mathbb{E} \sup_{0 \leq t \leq T} |x_{\varepsilon_1}(t) - x(t)|^2 \\
& \leq 3T \mathbb{E} \int_0^T |f(t, \psi_{\varepsilon_1}(t)) - f(t)|^2 dt \\
& + 12 \mathbb{E} \int_0^T |\sigma_1(t, \psi_{\varepsilon_1}(t)) - \sigma_1(t)|^2 dt \\
& + 12 \mathbb{E} \int_0^T |\sigma_2(t, \psi_{\varepsilon_1}(t)) - \sigma_2(t)|^2 dt \\
& \leq C_T \mathbb{E} \int_0^T |x_{\varepsilon_1}(t) - x(t)|^2 dt \\
& + \varepsilon_1^2 C_T \mathbb{E} \int_0^T |v_1(t) - u_1(t)|^2 dt,
\end{aligned}$$

where $C_T > 0$ is a constant only depending on $T > 0$ and the Lipschitz coefficients of f , σ_1 and σ_2 . From Gronwall's inequality we get the desired result. \square

Lemma 3.2. *Under Assumption (A1), it yields*

$$\begin{aligned}
\lim_{\varepsilon_1 \rightarrow 0} \mathbb{E} \sup_{0 \leq t \leq T} |\bar{x}_{\varepsilon_1}(t)|^2 &= 0, \\
\lim_{\varepsilon_2 \rightarrow 0} \mathbb{E} \sup_{0 \leq t \leq T} |\bar{x}_{\varepsilon_2}(t)|^2 &= 0.
\end{aligned} \tag{8}$$

Proof. Without loss of generality, we prove the first result of (8) and the latter one can be similarly derived. For $g = f, \sigma_1, \sigma_2$, we set

$$\begin{aligned}
a_g^1(t) &= \int_0^1 g_x(t, \phi_{\varepsilon_1}^\lambda(t)) d\lambda, \\
a_g^2(t) &= \int_0^1 g_{\bar{x}}(t, \phi_{\varepsilon_1}^\lambda(t)) d\lambda, \\
a_g^3(t) &= \int_0^1 (g_x(t, \phi_{\varepsilon_1}^\lambda(t)) - g_x(t)) d\lambda \cdot x^1(t) \\
&+ \int_0^1 (g_{\bar{x}}(t, \phi_{\varepsilon_1}^\lambda(t)) - g_{\bar{x}}(t)) d\lambda \cdot \mathbb{E}x^1(t) \\
&+ \int_0^1 (g_{v_1}(t, \phi_{\varepsilon_1}^\lambda(t)) - g_{v_1}(t)) d\lambda \cdot (v_1(t) - u_1(t)).
\end{aligned}$$

Due to Assumption (A1), a_g^1 and a_g^2 are both uniformly bounded and $\lim_{\varepsilon_1 \rightarrow 0} \mathbb{E} \left(\sup_{0 \leq t \leq T} |a_g^3(t)|^2 \right) = 0$. Then we have

$$\begin{cases} d\bar{x}_{\varepsilon_1}(t) = [a_f^1(t)\bar{x}_{\varepsilon_1}(t) + a_f^2(t)\mathbb{E}\bar{x}_{\varepsilon_1}(t) + a_f^3(t)]dt \\ \quad + [a_{\sigma_1}^1(t)\bar{x}_{\varepsilon_1}(t) + a_{\sigma_1}^2(t)\mathbb{E}\bar{x}_{\varepsilon_1}(t) + a_{\sigma_1}^3(t)]dw_1(t) \\ \quad + [a_{\sigma_2}^1(t)\bar{x}_{\varepsilon_1}(t) + a_{\sigma_2}^2(t)\mathbb{E}\bar{x}_{\varepsilon_1}(t) + a_{\sigma_2}^3(t)]dw_2(t), \\ \bar{x}_{\varepsilon_1}(0) = 0. \end{cases}$$

Applying Itô's formula to $|\bar{x}_{\varepsilon_1}(t)|^2$ and Assumption (A1), we have

$$\begin{aligned}
\mathbb{E} \left[\sup_{0 \leq t \leq T} |\bar{x}_{\varepsilon_1}(t)|^2 \right] &\leq C \mathbb{E} \int_0^T |\bar{x}_{\varepsilon_1}(t)|^2 dt \\
&+ C \mathbb{E} \left[\sup_{0 \leq t \leq T} (|a_f^3(t)|^2 + |a_{\sigma}^3(t)|^2 + |a_{\sigma_2}^3(t)|^2) \right].
\end{aligned}$$

Then we can get the first convergence result of (8) from Gronwall's inequality. \square

Since $(u_1(\cdot), u_2(\cdot))$ is the Nash equilibrium point, then it follows that

$$\varepsilon_1^{-1} [J_1(u_{\varepsilon_1}(\cdot), u_2(\cdot)) - J_1(u_1(\cdot), u_2(\cdot))] \geq 0 \tag{9}$$

and

$$\varepsilon_2^{-1} [J_2(u_1(\cdot), u_{\varepsilon_2}(\cdot)) - J_2(u_1(\cdot), u_2(\cdot))] \geq 0. \tag{10}$$

Lemma 3.3. *Let Assumptions (A1) and (A2) hold. Then the following variational inequality holds for $i=1, 2$:*

$$\begin{aligned}
& \mathbb{E} \int_0^T \left[l_{ix}(t)x^i(t) + l_{i\bar{x}}(t)\mathbb{E}x^i(t) + l_{iv_i}(t)(v_i(t) - u_i(t)) \right] dt \\
& + \mathbb{E}[\varphi_{ix}(x(T), \mathbb{E}x(T))x^i(T) + \varphi_{i\bar{x}}(x(T), \mathbb{E}x(T))\mathbb{E}x^i(T)] \\
& \geq 0.
\end{aligned} \tag{11}$$

Proof. We firstly prove (11) holds for $i = 1$ and the another case can be similarly derived. From (9), it yields

$$\begin{aligned}
0 &\leq \varepsilon_1^{-1} [J_1(u_{\varepsilon_1}(\cdot), u_2(\cdot)) - J_1(u_1(\cdot), u_2(\cdot))] \\
&= \varepsilon_1^{-1} \mathbb{E} \int_0^T [l_1(t, \psi_{\varepsilon_1}(t)) - l_1(t)] dt \\
&\quad + \varepsilon_1^{-1} \mathbb{E} [\varphi_1(x_{\varepsilon_1}(T), \mathbb{E}x_{\varepsilon_1}(T)) - \varphi_1(x(T), \mathbb{E}x(T))] \\
&= I_1 + I_2.
\end{aligned}$$

From Assumptions (A1), (A2) and Lemma 3.2, we derive

$$\begin{aligned}
I_1 &= \mathbb{E} \int_0^T \left[\int_0^1 l_{1x}(t, \phi_{\varepsilon_1}^\lambda(t)) d\lambda \cdot (x^1(t) + \bar{x}_{\varepsilon_1}(t)) \right. \\
&\quad + \int_0^1 l_{1\bar{x}}(t, \phi_{\varepsilon_1}^\lambda(t)) d\lambda \cdot \mathbb{E}(x^1(t) + \bar{x}_{\varepsilon_1}(t)) \\
&\quad + \left. \int_0^1 l_{1v_1}(t, \phi_{\varepsilon_1}^\lambda(t)) d\lambda \cdot (v_1(t) - u_1(t)) \right] dt \\
&\longrightarrow \mathbb{E} \int_0^T \left[l_{1x}(t)x^1(t) + l_{1\bar{x}}(t)\mathbb{E}x^1(t) \right. \\
&\quad \left. + l_{1v_1}(t)(v_1(t) - u_1(t)) \right] dt,
\end{aligned} \tag{12}$$

$$\begin{aligned}
I_2 &= \mathbb{E} \left[\int_0^1 \varphi_{1x}(\psi_{\varepsilon_1}^\lambda(T)) d\lambda \cdot (\bar{x}_{\varepsilon_1}(T) + x^1(T)) \right. \\
&\quad + \left. \int_0^1 \varphi_{1\bar{x}}(\psi_{\varepsilon_1}^\lambda(T)) d\lambda \cdot \mathbb{E}(\bar{x}_{\varepsilon_1}(T) + x^1(T)) \right] \\
&\longrightarrow \mathbb{E} \left[\varphi_{1x}(x(T), \mathbb{E}x(T))x^1(T) \right. \\
&\quad \left. + \varphi_{1\bar{x}}(x(T), \mathbb{E}x(T))\mathbb{E}x^1(T) \right].
\end{aligned} \tag{13}$$

Combining (12) with (13), the inequality (11) follows for $i = 1$. \square

Next, we define the *Hamiltonian function* $H_i : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \times U_1 \times U_2 \times \mathbb{R}^n \times \mathbb{R}^{n \times d_1} \times \mathbb{R}^{n \times d_2} \rightarrow \mathbb{R}$ as follows:

$$\begin{aligned} H_i(t, a, \tilde{a}, v_1, v_2, q_i, k_{i1}, k_{i2}) \\ \triangleq \langle q_i, f(a, \tilde{a}, v_1, v_2) \rangle + \langle k_{i1}, \sigma_1(a, \tilde{a}, v_1, v_2) \rangle \\ + \langle k_{i2}, \sigma_2(a, \tilde{a}, v_1, v_2) \rangle + l_i(a, \tilde{a}, v_1, v_2), \end{aligned}$$

and denote $H_i(t) = H_i(t, x(t), \mathbb{E}x(t), u_1(t), u_2(t), q_i(t), k_{i1}(t), k_{i2}(t))$, $i = 1, 2$. Let us consider the following adjoint BSDE of mean-field type

$$\begin{cases} -dq_i(t) = [f_x^*(t)q_i(t) + \mathbb{E}(f_{\tilde{x}}^*(t)q_i(t)) + \sigma_{1x}^*(t)k_{i1}(t) \\ + \mathbb{E}(\sigma_{1\tilde{x}}^*(t)k_{i1}(t)) + \sigma_{2x}^*(t)k_{i2}(t) \\ + \mathbb{E}(\sigma_{2\tilde{x}}^*(t)k_{i2}(t)) + l_{ix}^*(t) + \mathbb{E}l_{i\tilde{x}}^*(t)]dt \\ - k_{i1}(t)dw_1(t) - k_{i2}(t)dw_2(t), \\ q_i(T) = \varphi_{ix}^*(x(T), \mathbb{E}x(T)) + \mathbb{E}\varphi_{i\tilde{x}}^*(x(T), \mathbb{E}x(T)), \end{cases} \quad (14)$$

which coupled with (2) constitutes a forward-backward SDE (FBSDE) of mean-field type.

In the sequel, we state necessary conditions of Nash equilibrium points, i.e. stochastic maximum principle as follows:

Theorem 3.1 (Maximum Principle). *Suppose (A1) and (A2) hold. Let $(u_1(\cdot), u_2(\cdot))$ be a Nash equilibrium point of Problem (MF) with the corresponding solutions $x(\cdot)$ and $(q_i(\cdot), k_{i1}(\cdot), k_{i2}(\cdot))$ of (2) and (14). Then it follows that*

$$\mathbb{E}[H_{1v_1}(t)|\mathcal{G}_t^1](v_1 - u_1(t)) \geq 0 \quad (15)$$

and

$$\mathbb{E}[H_{2v_2}(t)|\mathcal{G}_t^2](v_2 - u_2(t)) \geq 0, \quad (16)$$

$dtd\mathbb{P} - a.e.,$ for any $v_1 \in U_1$ and $v_2 \in U_2$.

Proof. We firstly prove (15). Applying Itô's formula to $\langle x^1(t), q_1(t) \rangle$, for any $v_1(\cdot) \in \mathcal{U}_1$ we obtain

$$\begin{aligned} & \mathbb{E}[\varphi_{1x}(x(T), \mathbb{E}x(T))x^1(T) + \varphi_{1\tilde{x}}(x(T), \mathbb{E}x(T))\mathbb{E}x^1(T)] \\ &= \mathbb{E} \int_0^T \left[-l_{1x}(t)x^1(t) - l_{1\tilde{x}}(t)\mathbb{E}x^1(t) \right. \\ & \quad \left. - l_{1v_1}(t)(v_1(t) - u_1(t)) \right] dt \\ & \quad + \mathbb{E} \int_0^T H_{1v_1}(t)(v_1(t) - u_1(t)) dt. \end{aligned} \quad (17)$$

This together with the variational inequality (11) derives

$$\begin{aligned} & \mathbb{E} \int_0^T \mathbb{E}[H_{1v_1}(t)|\mathcal{G}_t^1](v_1(t) - u_1(t)) dt \\ &= \mathbb{E} \int_0^T H_{1v_1}(t)(v_1(t) - u_1(t)) dt \geq 0, \end{aligned} \quad (18)$$

for all $v_1(\cdot) \in \mathcal{U}_1$, which implies (15) holds. By the similar method above, (16) is also true. \square

3.2 Sufficient Conditions

In what follows, we proceed to establish the sufficient conditions of Nash equilibrium points (also called verification theorem).

Theorem 3.2 (Verification Theorem). *Let (A1) and (A2) hold. Let $(u_1(\cdot), u_2(\cdot)) \in \mathcal{U}_1 \times \mathcal{U}_2$ with the corresponding solutions x and (q_i, k_{i1}, k_{i2}) to the equations (2) and (14). Suppose $H_i(t, a, \tilde{a}, v_1, v_2, q_i(t), k_{i1}(t), k_{i2}(t))$ and φ_i are convex with respect to (a, \tilde{a}, v_i) . Moreover,*

$$\begin{aligned} & \mathbb{E}[H_1(t, x(t), \mathbb{E}x(t), u_1(t), u_2(t), q_1(t), k_{11}(t), \\ & k_{12}(t))|\mathcal{G}_t^1] = \inf_{v_1 \in U_1} \mathbb{E}[H_1(t, x(t), \mathbb{E}x(t), v_1, u_2(t), \\ & q_1(t), k_{11}(t), k_{12}(t))|\mathcal{G}_t^1], \end{aligned} \quad (19)$$

$$\begin{aligned} & \mathbb{E}[H_2(t, x(t), \mathbb{E}x(t), u_1(t), u_2(t), q_2(t), k_{21}(t), \\ & k_{22}(t))|\mathcal{G}_t^2] = \inf_{v_2 \in U_2} \mathbb{E}[H_2(t, x(t), \mathbb{E}x(t), u_1(t), v_2, \\ & q_2(t), k_{21}(t), k_{22}(t))|\mathcal{G}_t^2]. \end{aligned} \quad (20)$$

Then $(u_1(\cdot), u_2(\cdot))$ is a Nash equilibrium point of Problem (MF).

Proof. Let $v_i(\cdot) \in \mathcal{U}_i, i = 1, 2$. We denote by x^{v_1} and x the solutions to (2) associated with the admissible controls (v_1, u_2) and (u_1, u_2) , respectively. We set

$$\begin{aligned} & H_1(t) = H_1(t, x(t), \mathbb{E}x(t), u_1(t), u_2(t), q_1(t), k_{11}(t), \\ & k_{12}(t)), \quad H_1^{v_1}(t) = H_1(t, x^{v_1}(t), \mathbb{E}x^{v_1}(t), v_1(t), u_2(t), \\ & q_1(t), k_{11}(t), k_{12}(t)), \quad g(t) = g(t, x(t), \mathbb{E}x(t), u_1(t), u_2(t)), \\ & g^{v_1}(t) = g(t, x^{v_1}(t), \mathbb{E}x^{v_1}(t), v_1(t), u_2(t)), \quad g = f, \sigma_1, \sigma_2, l_1. \end{aligned}$$

By virtue of the convexity of φ_1 , we have for any $v_1(\cdot) \in \mathcal{U}_1$

$$J_1(v_1(\cdot), u_2(\cdot)) - J_1(u_1(\cdot), u_2(\cdot)) \geq I_1 + I_2 \quad (21)$$

with

$$\begin{aligned} I_1 &= \mathbb{E}[\varphi_{1x}(x(T), \mathbb{E}x(T))(x^{v_1}(T) - x(T)) \\ & \quad + \varphi_{1\tilde{x}}(x(T), \mathbb{E}x(T))\mathbb{E}(x^{v_1}(T) - x(T))], \\ I_2 &= \mathbb{E} \int_0^T (l_1^{v_1}(t) - l_1(t)) dt \\ &= \mathbb{E} \int_0^T [H_1^{v_1}(t) - H_1(t) - \langle q_1(t), f^{v_1}(t) - f(t) \rangle \\ & \quad - \langle k_{11}(t), \sigma_1^{v_1}(t) - \sigma_1(t) \rangle \\ & \quad - \langle k_{12}(t), \sigma_2^{v_1}(t) - \sigma_2(t) \rangle] dt. \end{aligned} \quad (22)$$

Applying Itô's formula to $\langle q_1(t), x^{v_1}(t) - x(t) \rangle$, we have

$$\begin{aligned} I_1 &= \mathbb{E} \int_0^T [\langle q_1(t), f^{v_1}(t) - f(t) \rangle \\ & \quad + \langle k_{11}(t), \sigma_1^{v_1}(t) - \sigma_1(t) \rangle + \langle k_{12}(t), \sigma_2^{v_1}(t) - \sigma_2(t) \rangle \\ & \quad - \langle H_{1x}^*(t), x^{v_1}(t) - x(t) \rangle \\ & \quad - \langle \mathbb{E}H_{1\tilde{x}}^*(t), x^{v_1}(t) - x(t) \rangle] dt. \end{aligned} \quad (23)$$

Substituting (22) and (23) into (21) and applying the convexity of H_1 , we get

$$\begin{aligned}
& J_1(v_1(\cdot), u_2(\cdot)) - J_1(u_1(\cdot), u_2(\cdot)) \\
& \geq \mathbb{E} \int_0^T \left(H_1^{v_1}(t) - H_1(t) - \langle H_{1x}^*(t), x^{v_1}(t) - x(t) \rangle \right. \\
& \quad \left. - \langle \mathbb{E} H_{1x}^*(t), x^{v_1}(t) - x(t) \rangle \right) dt \\
& \geq \mathbb{E} \int_0^T H_{1v_1}(t) (v_1(t) - u_1(t)) dt \\
& = \mathbb{E} \int_0^T \mathbb{E} [H_{1v_1}(t) | \mathcal{G}_t^1] (v_1(t) - u_1(t)) dt. \quad (24)
\end{aligned}$$

Condition (19) implies $\mathbb{E} [H_{1v_1}(t) | \mathcal{G}_t^1] (v_1(t) - u_1(t)) \geq 0$, $dt d\mathbb{P} - a.e.$ on $[0, T]$, which derives

$$J_1(v_1(\cdot), u_2(\cdot)) - J_1(u_1(\cdot), u_2(\cdot)) \geq 0.$$

Similarly, we can also derive $J_2(u_1(\cdot), v_2(\cdot)) - J_2(u_1(\cdot), u_2(\cdot)) \geq 0$. The proof is completed. \square

4 LQ Example

In this section, we work out an LQ example to illustrate the theoretical result. Without loss of generality, we only consider the following case: $n = d_1 = d_2 = 1, b_1(t)b_2(t) \neq 0, t \in [0, T]$. Throughout this section, we assume additional condition.

$$(A3) \quad m_1^{-1}(t)b_1^2(t) = m_2^{-1}(t)b_2^2(t), t \in [0, T].$$

Example 4.1. Consider the system of linear mean-field SDE

$$\begin{cases} dx^{v_1, v_2}(t) = [a(t)x^{v_1, v_2}(t) + \bar{a}(t)\mathbb{E}x^{v_1, v_2}(t) \\ \quad + b_1(t)v_1(t) + b_2(t)v_2(t)] dt \\ \quad + c_1(t)dw_1(t) + c_2(t)dw_2(t), \\ x^{v_1, v_2}(0) = x_0, \end{cases} \quad (25)$$

with the quadratic cost functional

$$\begin{aligned}
J_i(v_1(\cdot), v_2(\cdot)) &= \frac{1}{2} \mathbb{E} \left[\int_0^T \left(g_i(t)(x^{v_1, v_2}(t))^2 \right. \right. \\
&\quad \left. \left. + \bar{g}_i(t)(\mathbb{E}x^{v_1, v_2}(t))^2 + m_i(t)(v_i(t))^2 \right) dt \right. \\
&\quad \left. + h_i(x^{v_1, v_2}(T))^2 + \bar{h}_i(\mathbb{E}x^{v_1, v_2}(T))^2 \right] \quad (26)
\end{aligned}$$

and the information available to two players $\mathcal{G}_t^1 = \mathcal{G}_t^1 = \mathcal{F}_t^{w_1} = \sigma\{w_1(s), 0 \leq s \leq t\}$.

Here, all coefficients with respect to t in (25) and (26) are deterministic and uniformly bounded. In addition, g_i and \bar{g}_i are non-negative, h_i and \bar{h}_i are non-negative constants, and m_i is positive. The set of admissible controls for Player i is defined by

$$\begin{aligned}
\mathcal{U}_i &= \{v_i(\cdot) \mid v_i(\cdot) \text{ is an } \mathbb{R}\text{-valued } \mathcal{F}_t^{w_1}\text{-adapted process} \\
&\text{satisfying } \mathbb{E} \int_0^T v_i^2(t) dt < \infty\}, i = 1, 2. \quad (27)
\end{aligned}$$

Then the unique Nash equilibrium point is denoted by

$$\begin{cases} u_1(t) = -m_1^{-1}(t)b_1(t)(\tau_1(t)\hat{x}(t) + \delta_1(t)\mathbb{E}x(t)), \\ u_2(t) = -m_2^{-1}(t)b_2(t)(\tau_2(t)\hat{x}(t) + \delta_2(t)\mathbb{E}x(t)), \end{cases} \quad (28)$$

where $\mathbb{E}x$, (τ_1, τ_2) , (δ_1, δ_2) and \hat{x} are determined by (38), (45), (46) and (47), respectively.

Proof. We shall complete the proof by two parts.

Part I. We first need to prove the unique Nash equilibrium point can be represented by

$$\begin{cases} u_1(t) = -m_1^{-1}(t)b_1(t)\hat{q}_1(t), \\ u_2(t) = -m_2^{-1}(t)b_2(t)\hat{q}_2(t), \end{cases} \quad (29)$$

where x , (q_1, k_{11}, k_{12}) and (q_2, k_{21}, k_{22}) are the unique solution of the following coupled FBSDE of mean-field type

$$\begin{cases} dx(t) = [a(t)x(t) + \bar{a}(t)\mathbb{E}x(t) - m_1^{-1}(t)b_1^2(t)\hat{q}_1(t) \\ \quad - m_2^{-1}(t)b_2^2(t)\hat{q}_2(t)] dt + c_1(t)dw_1(t) + c_2(t)dw_2(t), \\ x(0) = x_0, \\ -dq_1(t) = [a(t)q_1(t) + \bar{a}(t)\mathbb{E}q_1(t) + g_1(t)x(t) \\ \quad + \bar{g}_1(t)\mathbb{E}x(t)] dt - k_{11}(t)dw_1(t) - k_{12}(t)dw_2(t), \\ q_1(T) = h_1x(T) + \bar{h}_1\mathbb{E}x(T), \end{cases} \quad (30)$$

and

$$\begin{cases} -dq_2(t) = [a(t)q_2(t) + \bar{a}(t)\mathbb{E}q_2(t) + g_2(t)x(t) \\ \quad + \bar{g}_2(t)\mathbb{E}x(t)] dt - k_{21}(t)dw_1(t) - k_{22}(t)dw_2(t), \\ q_2(T) = h_2x(T) + \bar{h}_2\mathbb{E}x(T). \end{cases} \quad (32)$$

Here we denote by $\hat{p}(t)$ the mathematical expectation of $p(t)$ with respect to $\mathcal{F}_t^{w_1}$, i.e., $\hat{p}(t) \triangleq \mathbb{E}[p(t) | \mathcal{F}_t^{w_1}]$, $p = q_1, q_2, k_{11}, k_{21}, x$. The rest of Part I is divided into the following two steps.

Step (i) (u_1, u_2) of the form (29) is the Nash equilibrium point indeed.

We first write down the Hamiltonian function

$$\begin{aligned}
H_i(t, x, \tilde{x}, v_1, v_2, q_i, k_{i1}, k_{i2}) \\
&\triangleq q_i(a(t)x + \bar{a}(t)\tilde{x} + b_1(t)v_1 + b_2(t)v_2) + k_{i1}c_1 \\
&\quad + k_{i2}c_2 + \frac{1}{2}(g_ix^2 + \bar{g}_i\tilde{x}^2 + m_iv_i^2). \quad (33)
\end{aligned}$$

Applying Theorem 3.1, we derive the candidate Nash equilibrium point of the form (29) and the coupled FBSDE of mean-field type (30)-(32). We can check that $\varphi_i(x, \tilde{x}) = \frac{1}{2}(h_ix^2 + \bar{h}_i\tilde{x}^2)$ and $H_i(t, x, \tilde{x}, v_1, v_2, q_i, k_{i1}, k_{i2})$ in (33) satisfy the conditions in Theorem 3.2. Therefore, $(u_1(\cdot), u_2(\cdot))$ of the form (29) is the Nash equilibrium point indeed.

Based on the arguments in Step (i), we conclude that the existence and uniqueness of the Nash equilibrium points are equivalent to the existence and uniqueness of the solutions to (30)-(32).

Step (ii) The solutions of (30)-(32) are existent and unique.

Taking mathematical expectation on both sides of (30)-(32), we have the following forward-backward ordinary equations

$$\begin{cases} d\mathbb{E}x(t) = [(a(t) + \bar{a}(t))\mathbb{E}x(t) \\ - m_1^{-1}(t)b_1^2(t)\mathbb{E}q_1(t) - m_2^{-1}(t)b_2^2(t)\mathbb{E}q_2(t)]dt, \\ \mathbb{E}x(0) = x_0, \end{cases} \quad (34)$$

$$\begin{cases} -d\mathbb{E}q_1(t) = [(a(t) + \bar{a}(t))\mathbb{E}q_1(t) \\ + (g_1(t) + \bar{g}_1(t))\mathbb{E}x(t)]dt, \\ \mathbb{E}q_1(T) = (h_1 + \bar{h}_1)\mathbb{E}x(T), \end{cases} \quad (35)$$

and

$$\begin{cases} -d\mathbb{E}q_2(t) = [(a(t) + \bar{a}(t))\mathbb{E}q_2(t) \\ + (g_2(t) + \bar{g}_2(t))\mathbb{E}x(t)]dt, \\ \mathbb{E}q_2(T) = (h_2 + \bar{h}_2)\mathbb{E}x(T). \end{cases} \quad (36)$$

Applying the method as shown in Chang and Xiao [7] and Assumption (A3), we can prove there exists a unique solution $(\mathbb{E}x(t), \mathbb{E}q_1(t), \mathbb{E}q_2(t))$ to (34)-(36) with the relations as follows:

$$\mathbb{E}q_i(t) = \alpha_i(t)\mathbb{E}x(t), \quad i = 1, 2, \quad (37)$$

$$\mathbb{E}x(t) = x_0 e^{\int_0^t [a(s) + \bar{a}(s) - m_1^{-1}(s)b_1^2(s)(\alpha_1(s) + \alpha_2(s))]ds}, \quad (38)$$

where (α_1, α_2) is the unique solution of the following Riccati equations

$$\begin{aligned} \dot{\alpha}_1 + 2(a + \bar{a})\alpha_1 - m_1^{-1}b_1^2\alpha_1^2 - m_2^{-1}b_2^2\alpha_1\alpha_2 \\ + g_1 + \bar{g}_1 = 0, \end{aligned} \quad (39)$$

$$\begin{aligned} \dot{\alpha}_2 + 2(a + \bar{a})\alpha_2 - m_1^{-1}b_1^2\alpha_1\alpha_2 - m_2^{-1}b_2^2\alpha_2^2 \\ + g_2 + \bar{g}_2 = 0, \end{aligned} \quad (40)$$

subject to $\alpha_i(T) = h_i + \bar{h}_i$.

Substituting (37) into (31) and (32), taking conditional mathematical expectation on both sides of (30)-(32) with respect to $\mathcal{F}_t^{w_1}$ and applying Lemma 5.4 in Xiong [14], we have

$$\begin{cases} d\hat{x}(t) = [a(t)\hat{x}(t) + \bar{a}(t)\mathbb{E}x(t) - m_1^{-1}(t)b_1^2(t)\hat{q}_1(t) \\ - m_2^{-1}(t)b_2^2(t)\hat{q}_2(t)]dt + c_1(t)dw_1(t), \\ \hat{x}(0) = x_0, \end{cases} \quad (41)$$

$$\begin{cases} -d\hat{q}_1(t) = [a(t)\hat{q}_1(t) + g_1(t)\hat{x}(t) \\ + (\bar{a}(t)\alpha_1(t) + \bar{g}_1(t))\mathbb{E}x(t)]dt - \hat{k}_{11}(t)dw_1(t), \\ \hat{q}_1(T) = h_1\hat{x}(T) + \bar{h}_1\mathbb{E}x(T), \end{cases} \quad (42)$$

$$\begin{cases} -d\hat{q}_2(t) = [a(t)\hat{q}_2(t) + g_2(t)\hat{x}(t) \\ + (\bar{a}(t)\alpha_2(t) + \bar{g}_2(t))\mathbb{E}x(t)]dt - \hat{k}_{21}(t)dw_1(t), \\ \hat{q}_2(T) = h_2\hat{x}(T) + \bar{h}_2\mathbb{E}x(T), \end{cases} \quad (43)$$

which constitute a kind of fully coupled forward-backward stochastic filtering equations of mean-field type and exist the unique solution $(\hat{x}, \hat{q}_1, \hat{k}_{11}, \hat{q}_2, \hat{k}_{21})$.

Similarly, based on the known $\mathbb{E}x, \mathbb{E}q_1, \mathbb{E}q_2, \hat{q}_1$ and \hat{q}_2 , there also exists the unique solution $(x, (q_1, k_{11}, k_{12}), (q_2, k_{21}, k_{22}))$ to (30)-(32).

Part 2. We need to verify the feedback form of the Nash equilibrium point in (29) is represented by (28).

Based on the terminal conditions in (42) and (43), we set

$$\hat{q}_i(t) = \tau_i(t)\hat{x}(t) + \delta_i(t)\mathbb{E}x(t), \quad i = 1, 2, \quad (44)$$

subject to $\tau_i(T) = h_i$ and $\delta_i(T) = \bar{h}_i$. Applying Itô's formula to $\hat{q}_1(t)$ (resp. $\hat{q}_2(t)$) in (44) and comparing the coefficients of $\hat{x}(t)$ and $\mathbb{E}x(t)$ between it and (42) (resp. (43)), respectively, we get

$$\begin{cases} \dot{\tau}_1 + 2a\tau_1 - m_1^{-1}b_1^2\tau_1^2 - m_2^{-1}b_2^2\tau_1\tau_2 + g_1 = 0, \\ \dot{\tau}_2 + 2a\tau_2 - m_2^{-1}b_2^2\tau_2^2 - m_1^{-1}b_1^2\tau_1\tau_2 + g_2 = 0, \end{cases} \quad (45)$$

$$\begin{cases} \dot{\delta}_1 + (2a + \bar{a} - m_1^{-1}b_1^2\alpha_1 - m_2^{-1}b_2^2\alpha_2 - m_1^{-1}b_1^2\tau_1)\delta_1 \\ - m_2^{-1}b_2^2\tau_1\delta_2 + \bar{a}\tau_1 + \bar{a}\alpha_1 + \bar{g}_1 = 0, \\ \dot{\delta}_2 + (2a + \bar{a} - m_1^{-1}b_1^2\alpha_1 - m_2^{-1}b_2^2\alpha_2 - m_2^{-1}b_2^2\tau_2)\delta_2 \\ - m_1^{-1}b_1^2\tau_2\delta_1 + \bar{a}\tau_2 + \bar{a}\alpha_2 + \bar{g}_2 = 0, \end{cases} \quad (46)$$

with $\tau_i(T) = h_i$ and $\delta_i(T) = \bar{h}_i$. Applying the method as shown in Chang and Xiao [7] and Assumption (A3), there exist the unique solutions to (45) and (46).

Substituting (44) into (41), we can derive the explicit solution as follow:

$$\begin{aligned} \hat{x}(t) = x_0\Phi_0^t + \int_0^t \Phi_s^t c_1(s)dw_1(s) \\ + \int_0^t \Phi_s^t (\bar{a} - m_1^{-1}b_1^2\delta_1 - m_2^{-1}b_2^2\delta_2)(s)\mathbb{E}x(s)ds \end{aligned} \quad (47)$$

with $\Phi_s^t = \exp\{\int_s^t (a - m_1^{-1}b_1^2\tau_1 - m_2^{-1}b_2^2\tau_2)(r)dr\}$. The proof is completed. \square

5 Conclusion Remarks

In this paper, we study non-zero sum mean-field game with partial information and derive the stochastic maximum principle and verification theorem for the Nash equilibrium points. Compared with the existing literature, the contributions of this paper are:

- Partial information is more general case than complete information. The results partly generalize the related mean-field control or game problems with complete information (see e.g. [1, 2, 11, 15]);
- Mean-field-type forward-backward stochastic filtering equations are found, which enriches the theory of classical filtering;
- The unique Nash equilibrium point in the LQ example is represented in the feedback form of not only the optimal filtering but also the expected value of the system state, through the solutions of some Riccati equations.

In addition, since there are many partial information mean-field game problems in finance and economics, we hope the results have applications in these related areas.

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